Zero-Knowledge Against Quantum Attacks

John Watrous (2009)

Sebastian Verschoor

QIC710: Introduction to Quantum Information Processing
University of Waterloo

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Outline

Interactive proof systems

- A statement (x) is True iff $x \in L$ for some fixed language L
  - Example: L is the language of all pairs of graphs that are isomorphic, x is the pair (G_0, G_1)
- Proving as an interactive procedure
  - Prover (P) convinces Verifier (V) of the truth of some statement (x) by giving a proof/certificate/witness (w)
- (Optional) restrictions:
  - Verifier is modelled by a Turing machine
  - Verifier runs in polynomial time
    - Size of the proof $(|w|)$ is polynomial
  - Verifier might only accept with probability $\geq 2/3$

Zero-knowledge

- In general, a proof system is concerned about two things:
  - Completeness: if both parties play honest, will V accept?
  - Soundness: if P cheats, will V reject?
- Zero-knowledge handles the case in which the V cheats:
  - Zero-knowledge: the protocol asserts nothing but $x \in L$
  - Leakage includes:
    - V cannot convince a third party of $x \in L$
    - V cannot convince a third party of "P knows that $x \in L$"
    - V cannot convince a third party that any conversation between P and V took place at all
- How to prove "Zero-knowledgeness"?

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Zero-knowledge protocols

- An interactive protocol between P and V is *zero-knowledge* on L if for all possible (cheating) verifiers (V'):

$$\boxed{\text{View}_{P,V'} \text{ is approximable on } L' = \{(x, H) | x \in L \wedge |H| = |x|^c\}}$$

- *View* is all V' sees
  - Random bits
  - Messages from P
  - Additional helper data H
- A View is *approximable* if there exists an efficient Turing machine S that creates a distribution that is indistinguishable from the View.
  - S is called the *simulator*

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Defining indistinguishability

- Families of random variables $U : \{U(x)\}$ and $V : \{V(x)\}$
  - If there is no judge J that can tell if the sample came from $U(x)$ or $V(x)$, U and V are indistinguishable
- Types of indistinguishability:
  - **Perfect**: J gets arbitrary many samples
    - $U = V$
  - **Statistical**: J gets only polynomial many samples
    - Statistical difference between U and V is negligible
  - **Computational**: J gets only polynomial many samples and has only polynomial time to distinguish them
    - U and V cannot be distinguished by an efficient algorithm

Applications of zero-knowledge

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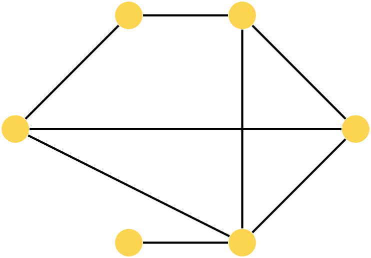
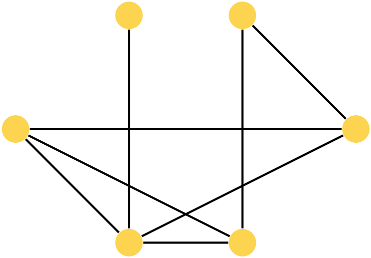
- Online authentication scheme
  - Client proves (in zero-knowledge) to a web-server that they know the password
  - (Note: this is not how the internet actually works: usually you just send a plaintext password)
- ZCASH

Example: graph isomorphism

Common input $x = (G_0, G_1)$:



G_0

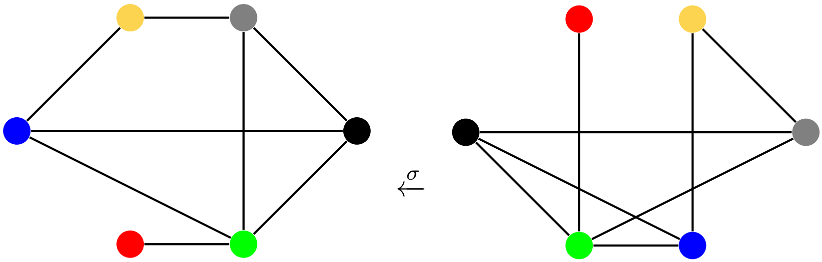


G_1

To prove: $G_0 \simeq G_1$

Example: graph isomorphism

Prover knows permutation σ such that $\sigma(G_1) = G_0$:



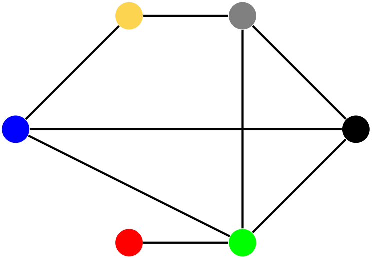
$G_0 = \sigma(G_1)$

G_1

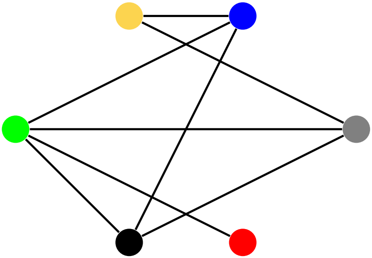
σ can only exist if $G_0 \simeq G_1$

Example: graph isomorphism

Prover chooses random permutation π and computes $H = \pi(G_0)$:



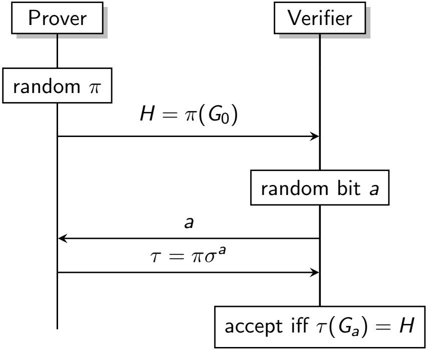
$\xrightarrow{\pi}$



$G_0 = \sigma(G_1)$ $\qquad$ $H = \pi(G_0) = \pi\sigma(G_1)$

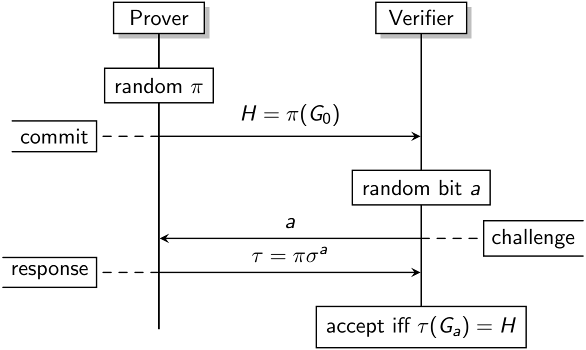
Prover sends H to Verifier

Example: graph isomorphism



- Repeat until V is satisfied
- To be a Zero-knowledge proof of $G_0 \simeq G_1$, we need to prove:
  - Completeness: $G_0 \simeq G_1 \Rightarrow \Pr[\text{accept}] = 1$
  - Soundness: $G_0 \not\simeq G_1 \Rightarrow \Pr[\text{reject}] = 1/2$
  - Zero-knowledge: does a simulator exist?

Example: graph isomorphism



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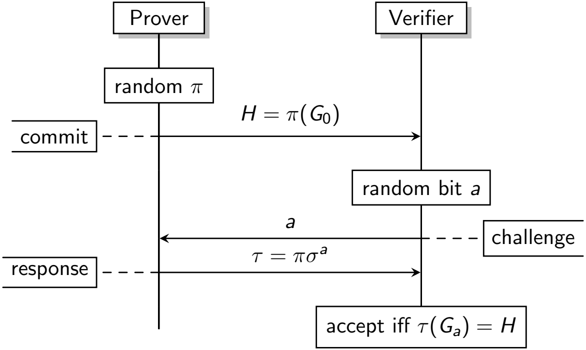

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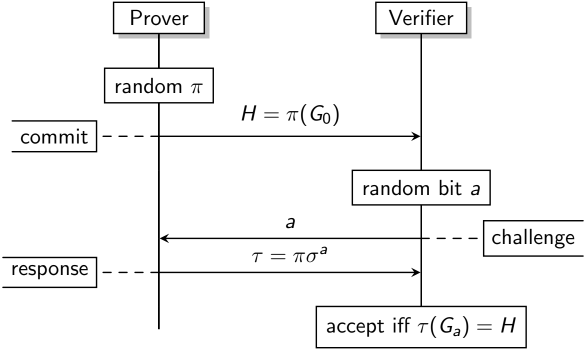

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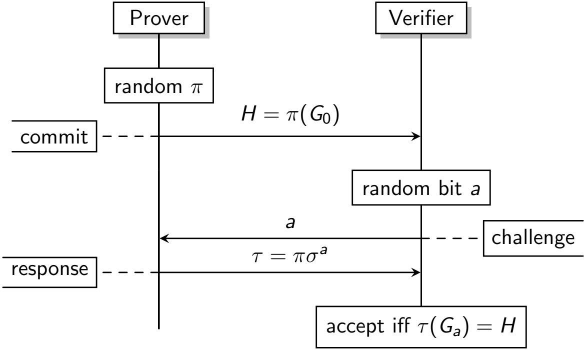

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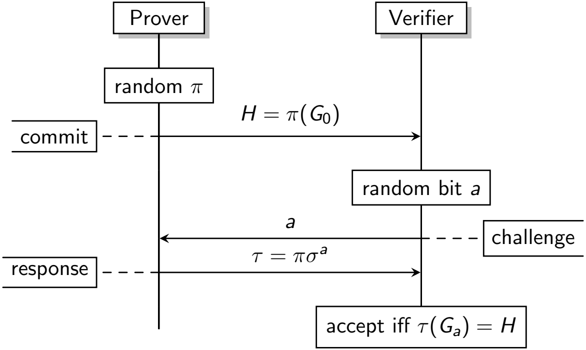

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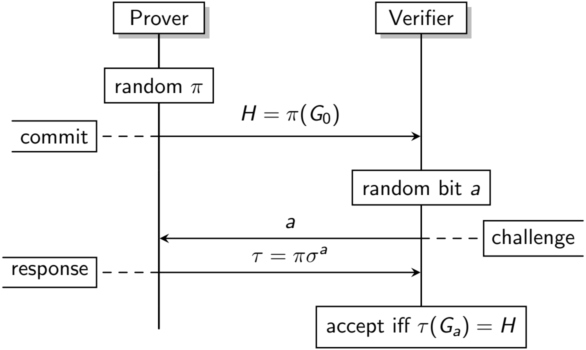

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Example: graph isomorphism

- Q: Does a simulator exist?
- A: Sure, just take out your time machine!
- Define a simulator $S^{V'}$ that uses V' as a subroutine

1. Pick a random permutation τ and bit a'
2. Send $H = \tau(G_{a'})$ to V'
3. V' replies with a
4. if $a' = a$: send τ
   else: go back in time (rewind V') and try again!

- Efficient: expected to rewind once
- $View_{P,V'} = View_{S^{V'},V'}$

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Example: graph isomorphism

- It's all software: we don't need a physical time machine
- $S^{V'}$ sets up V' in a virtual machine
- Before every iteration of the protocol: take a snapshot
- if $a' = a$: success
  else: restart from the snapshot and try again
- What have we achieved?
  - Any transcript that V' gives to J could have been created by $S^{V'}$, who does not have any knowledge
  - So no transcript can leak any knowledge

Quantum Attacks

What if cheating verifier V' has a quantum computer?

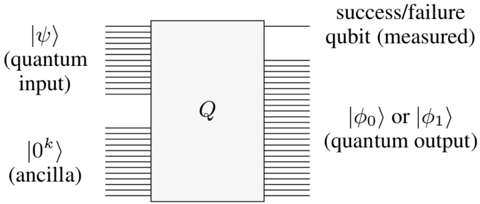
- V' could have auxiliary input entangled with qubits not accessible to V' or $S^{V'}$, but available to the judge
- Rewinding cannot be applied *generally*
  - Quantum information cannot be copied
  - Running V' might involve a irreversible measurement
  - Determining if a simulation was succesful *requires* a measurement

Quantum Zero-Knowledge

- Need to refine our notion of indistinguishability
- Instead of the View, we take a look at possible quantum *channels* that the cheating verifier can implement
- We use the Kitaev diamond norm distance between two channels Φ_0 and Φ_1:
  - $\frac{1}{2}\|\Phi_0 - \Phi_1\|_\diamond$
  - Describes the maximum bias with which a physical process can distinguish them
  - Covers distinguishing with entangled states
  - This is analogous to the trace distance between quantum *states*

We can't rewind in general. But we can if:

- Given a circuit Q of the form:



- In general, this circuit implements:

$$Q|\psi\rangle|0^k\rangle = \sqrt{p(\psi)}\,|0\rangle|\phi_0(\psi)\rangle + \sqrt{1-p(\psi)}\,|1\rangle|\phi_1(\psi)\rangle$$

- We can *rewind* if $p = p(\psi)$ is constant (independent of ψ).
- Goal: to get $|\phi_0(\psi)\rangle$ with probability arbitrary close to 1

Quantum rewinding lemma

- Getting $|\phi_0(\psi)\rangle$ from $|\psi\rangle$:
  1. Apply Q
  2. Repeat:
  3. Measure first output register
  4. If outcome is 1:
  5. Apply Q^{-1}
  6. Apply $U = 2|0^m\rangle\langle 0^m| - \mathbb{1}$ to ancilla
  7. Apply Q
  8. Output $|\phi\rangle$
- After applying Q, we get in state

$$Q|\psi\rangle|0^k\rangle = \sqrt{p}\,|0\rangle|\phi_0(\psi)\rangle + \sqrt{1-p}\,|1\rangle|\phi_1(\psi)\rangle$$

- If we measure 0, we are done! Else we apply

$$Q(I \otimes U)Q^{-1}|1\rangle|\phi(\psi)\rangle$$
$$= 2\sqrt{p(1-p)}\,|0\rangle|\phi_0(\psi)\rangle + (1-2p)\,|1\rangle|\phi_1(\psi)\rangle$$

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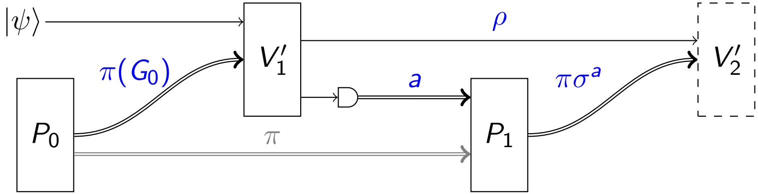
A cheating verifier has the following interaction:

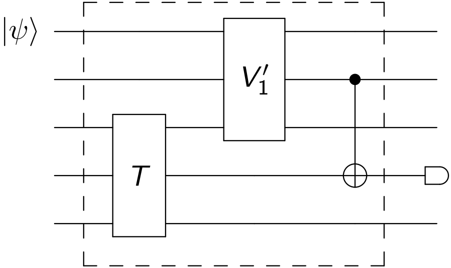


Where $\rho = \frac{1}{n!} \sum_{\pi \in S_n} \sum_{a \in \{0,1\}} M_{\pi(G_0),a} |\psi\rangle\langle\psi| M^*_{\pi(G_0),a}$

The channel Φ is then the tensor product of all registers in the view

Example: graph isomorphism

We simulate this using the following Q:



Where T creates a uniform superposition:

$$\frac{1}{\sqrt{2n!}} \sum_{\tau \in S_n} \sum_{b \in \{0,1\}} |\tau(G_b)\rangle \, |b\rangle \, |\tau\rangle$$

Example: graph isomorphism

- Works out to $p = 1/2$ with compatible states ϕ_0 and ϕ_1
- Applying the Quantum Rewinding lemma once to $|1\rangle\,|\phi_1(\psi)\rangle$
  $2\sqrt{p(1-p)}\,|0\rangle\,|\phi_0(\psi)\rangle + (1-2p)\,|1\rangle\,|\phi_1(\psi)\rangle$
  $= 2\sqrt{1/4}\,|0\rangle\,|\phi_0(\psi)\rangle + 0\,|1\rangle\,|\phi_1(\psi)\rangle$
  $= |0\rangle\,|\phi_0(\psi)\rangle$
- For graph isomorphism, we need to rewind (at most) once

Generalizing the results

- Relax the assumption that p is independent of $|\psi\rangle$
  - When $p(\psi)$ varies only by an exponentially small amount, we can still achieve statistical zero-knowledge
- The construction applies to all protocols of the form:
  1. P sends a message to V
     - Message could even be some quantum state
  2. V flips a fair coin and sends the result to P
  3. P responds with a second message
     - Message could even be some quantum state
- All problems in HVQSZK can be expressed in this form
  - HVQSZK = QSZK
- SZK $\subseteq$ HVQSZK
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- Complexity class results
  - QSZK is closed under complement
  - The "close quantum states" problem is complete for QSZK
  - QSZK $\subseteq$ QIP(2)
  - $\mathrm{QSZK}_{a,b} = \mathrm{QSZK}_{1,c}$ with c polynomialy small
  - Similar results for QZK and QPZK
- Non-interactive zero-knowledge proofs
  - Quantum non-interactive zero-knowledge proofs

Further reading

- Zero Knowledge Proofs: An illustrated primer – Matthew Green (2007) [link]
- Zero-Knowledge Against Quantum Attacks – John Watrous (2009) [link]
- Quantum Proofs – Thomas Vidick, John Watrous (2016) [link]
- Slides will be available at my website: [zeroknowledge.me]

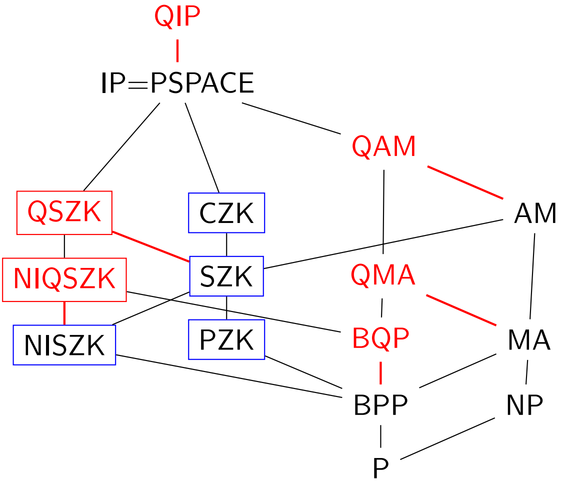
Thank you

Complexity classes

We can use the interactive games to define complexity classes

- **NP**: Non-deterministic Polynomial time
  - $L \in \mathbf{NP}$ if a short proof exists for an efficient verifier
    - Formally: there exist polynomials p, q and verifier V such that
      $\forall x \in L : \exists w : |w| = q(|x|) \wedge V(x, w) = 1$
      $\forall x \notin L : |w| = q(|x|) \Rightarrow V(x, w) = 0$
      $\forall x, w : V(x, w)$ runs in time $p(|x|)$
- **P**: Polynomial time
  - $L \in \mathbf{P}$ if an efficient verifier exists
    - Formally: There exists a polynomial p and verifier V such that
      $\forall x \in L : V(x, \emptyset) = 1$
      $\forall x \notin L : V(x, \emptyset) = 0$
      $\forall x : V(x, \emptyset)$ runs in time $p(|x|)$
- $\mathbf{P} \subseteq \mathbf{NP}$

- **BPP**: Bounded-error Probabilistic Polynomial time
  - $L \in$ **BPP** if an efficient *probabilistic* verifier exists
    $\forall x \in L : \Pr[V(x, \emptyset) = 1] \geq 2/3$
    $\forall x \notin L : \Pr[V(x, \emptyset) = 0] \leq 1/3$
- **MA**: Merlin-Arthur
  - Arthur is a **BPP** verifier
- **AM**: Arthur-Merlin
  - Arthur can send a message (challenge) to Merlin before Merlin provides a proof
- **IP**: Interactive Proof systems
  - Like **AM**, but allows many rounds interactions
- $\mathbf{P} \subseteq \mathbf{BPP} \subseteq \mathbf{MA} \subseteq \mathbf{AM} \subseteq \mathbf{IP}$


QIP
IP=PSPACE
QAM
QSZK
CZK
AM
NIQSZK
SZK
QMA
NISZK
PZK
BQP
MA
BPP
NP
P

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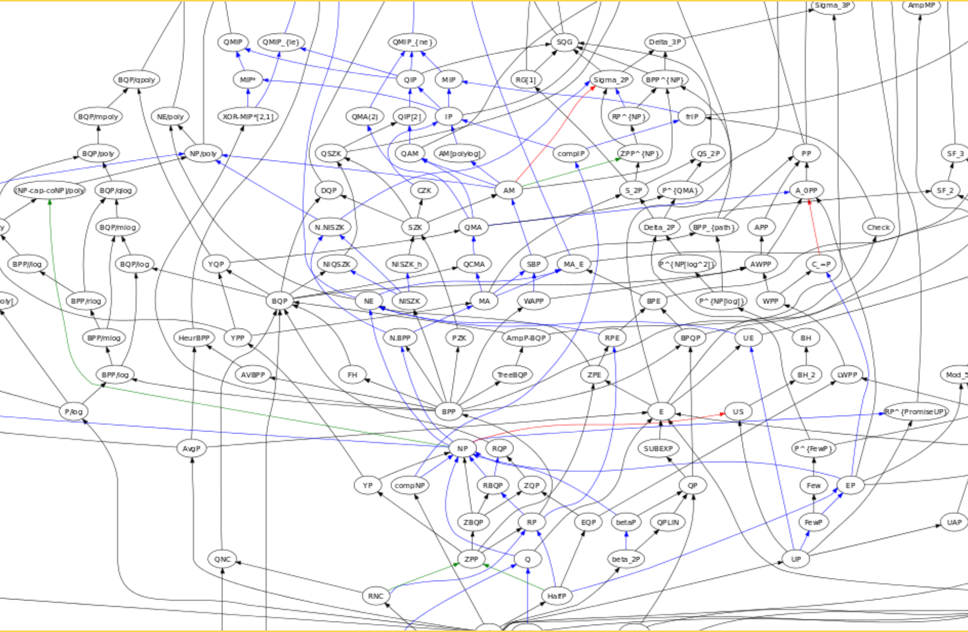

Complexity class relations